Aniqmas integral

1-mavzu. Aniqmas integral va uning xossalari

Reja

1. **Boshlang‘ich funksiya va uning xossasi.**
2. **Aniqmas integral va uning xossalari.**
3. **Asosiy integrallar jadvali.**

1. Boshlang‘ich funksiya va uning xossasi. Ma’lumki matematikada amallar juft-juft bo‘lib uchrab keladi. Jumladan, qo‘shish va ayirish, ko‘paytirish va bo‘lish, darajaga ko‘tarish va ildiz chiqarish va boshqalar. Funksiya hosilasini topishga yoki differensialash amaliga teskari amal bormikan degan tabiiy savol tug‘iladi.

Differensial hisobda funksiya berilgan bo‘lsa, uning hosilasini topishni qaradik. Haqiqatda ham fan va texnikaning bir qancha masalalarini hal etishda teskari masalani Yechishga to‘g‘ri keladiki, berilgan $f(x)$ funksiya uchun shunday, $F(x)$ funksiyani topish kerakki, uning hosilasi berilgan $f(x)$ funksiyaga teng bo‘lsin. Ma’lumki, bunday $F(x)$ funksiyaga berilgan $f(x)$ funksiyaning ***boshlang‘ich*** **(dastlabki)** ***funksiyasi*** deyiladi.

Masalan, $y = f(x) = x^4$ funksiyaning boshlanђich funksiyasi, $F(x) = \frac{x^5}{5}$ bo’ladi, chunki $F'(x) = (\frac{x^5}{5})' = x^4 = f(x)$ bo’ladi.

2. Aniqmas ***integral va uning xossalari.*** Ta’rif. $F(x)$ funksiya biror oraliqda $f(x)$ funksiyaning boshlang‘ich funksiyasi bo‘lsa, $F(x)+C$ (bunda C ixtiyoriy o‘zgarmas) funksiyalar to‘plami shu oraliqda $f(x)$ ***funksiyaning aniqmas integrali*** deyiladi va

$$\int f(x)dx = F(x) + C$$

bilan belgilanadi. Bu erda $f(x)$ integral ostidagi funksiya, $f(x)dx$ integral ostidagi ifoda, x integrallash o‘zgaruvchisi, $\int$ integral belgisi deyiladi.

Demak, $\int f(x)dx$ simvol, $f(x)$ funksiyaning hamma boshlang‘ich funksiyalari to‘plamini belgilaydi.

Berilgan funksiyaning aniqmas integralini topish amaliga integrallash deyiladi.

Aniqmas *<u>integralning xossalari</u>*:

1) aniqmas integralning hosilasi integral ostidagi funksiyaga, differensiali esa integral ostidagi ifodaga teng, ya’ni

$$\left(\int f(x)dx\right)' = f(x) \qquad \text{ва} \qquad d\int F(x)dx = F(x)dx;$$

2) biror funksiyaning hosilasidan hamda differensialidan aniqmas integral shu funksiya bilan ixtiyoriy o‘zgarmasning yig‘indisiga teng, ya’ni

$$f'(x)dx = f(x) + C \qquad \text{ва} \qquad \int dF(x) = F(x) + C.$$

Bu xossalar aniqmas integralning ta’rifidan bevosita kelib chiqadi. Haqiqatan, 1-xossadan $\left(\int f(x)dx\right)' = \left(F(x) + C\right)' = F'(x) + 0 = f(x)$ bo’ladi. (Qolganlarini keltirib chiqarish o‘quvchiga havola etiladi).

Bu xossalardan differensiallash va integrallash amallari o‘zaro teskari amallar ekanligini payqash mumkin.

3) o’zgarmas ko‘paytuvchini integral belgisi tashqarisiga chiqarish mumkin, ya’ni $K = const \neq 0$ bo‘lsa,

$$\int Kf(x)dx = K\int f(x)dx;$$

4) chekli sondagi funksiyalar algebraik yig‘indisining aniqmas integrali, shu funksiyalar aniqmas integrallarining algebraik yig‘indisiga teng, ya’ni

$$\int \left[f_1(x) + f_2(x) - f_3(x) \right] dx = \int f_1(x)dx + \int f_2(x)dx - \int f_3(x)\, dx.$$

3. **<u>Asosiy integrallar jadvali</u>.** Berilgan funksiyaga asosan uning boshlang‘ichini topish, berilgan funksiyani differensiallashga nisbatan ancha murakkabroq masaladir. Differensial hisobda asosiy elementar funksiyalarning, yig‘indining, ko‘paytmaning, bo‘linmaning hamda murakkab funksiyalarning hosilasini topishni o‘rgandik. Bu qoidalar istalgan elementar funksiyalarning hosilasini topishga imkon berdi. Elementar funksiyalarni integrallashda esa differensiallashdagidek umumiy qoidalar yo‘q. masalan, ikkita elementar funksiyalar boshlang‘ichlarining ma’lum bo‘lishiga qaramasdan, ular ko‘paytmasining, bo‘linmasining boshlang‘ichini topishda aniq bir qoida yo‘q.

Integrallashda integral ostidagi ifodaning muayyan berilishiga qarab, unga mos individual usullardan foydalanishga to‘g‘ri keladi. Boshqacha aytganda, integrallashda ancha kengroq fikr yuritish kerak bo‘ladi. Funksiyani integrallash ya’ni boshlang‘ich funksiyani topish metodlari bir qancha shunday usullarni ko‘rsatadiki, ular yordamida ko‘p hollarda maqsadga erishiladi.

Integrallashda maqsadga erishish uchun quyidagi **asosiy integrallar jadvalini** yoddan bilish zarur.

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1;$ 2) $\int dx = x + C;$ 3) $\int \frac{1}{x} dx = \ln|x| + C;$

4) $\int \sin x dx = -\cos x + C;$ 5) $\int \cos x dx = \sin x + C;$ 6) $\int e^x dx = e^x + C;$

7) $\int a^x dx = \frac{a^x}{\ln a} + C, \ (0 < a \neq 1);$ 8) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} arctg \frac{x}{a} + C;$

9) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C;$ 10) $\int \frac{1}{\cos^2 x} dx = tgx + C;$

$\int$

11) $\int \frac{1}{\sin^2 x} dx = -ctgx + C;$ 12) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \frac{x-a}{x+a} + C, \ a \neq 0;$

13) $\int \frac{dx}{\sqrt{x^2 - k}} = \ln x + \sqrt{x^2 - k} + C.$

Bu formulalarning to‘g‘riligini, tekshirish tengliklarning o‘ng tomonidagi ifodalar differensiali integral ostidagi ifodaga teng ekanligini ko‘rsatishdan iboratdir. Masalan,

$$d\left(\frac{x^n + 1}{n+1} + C\right) = \left(\frac{x^n + 1}{n+1} + C\right)' dx = \frac{(n+1)x^n}{n+1} dx = x^n dx.$$

Integrallashga bir necha misollar qaraymiz.

1-misol. $\int (x^3 + 5\sin x - 9)dx$ integralni hisoblang.

Yechish. Integralning 4 va 3 xossalariga asosan,

$$\int (x^3 + 5\sin x - 9)dx = \int x^3 \, dx + 5\int \sin x \, dx - 9\int dx$$

bo‘ladi. Asosiy integrallar jadvalidagi 1), 2), 4) formulalarga asosan,

$$\int x^3dx=\frac{x^4}{4}+C_1,\ \ 5\int \sin xdx=5(-\cos x+C_2),\ \ -9\int dx=-9(x+C_3).$$

Demak,

$$\int (x^3+5\sin x-9)dx=\frac{x^4}{4}-5\cos x-9x+(C_1+5C_2-9C_3)\,.$$

Yuqoridagi integralni hisoblashda har bir uchta integralda o‘zining ixtiyoriy o‘zgarmasini qo‘shdik, lekin oxirgi natijada bitta ixtiyoriy o‘zgarmasni qo‘shamiz, chunki C_1, C_2, C_3 ixtiyoriy o‘zgarmaslar bo‘lsa,

$C=C_1+5C_2-9C_3$ ham ixtiyoriy o‘zgarmas bo‘ladi, shuning uchun, oxirgi natijani quyidagicha yozamiz:

$$\int (x^3+5\sin x-9)dx=\frac{x^4}{4}-5\cos x-9x+C\,.$$

Integralning to‘g‘ri hisoblanganligini tekshirish uchun oxirgi tenglikning o‘ng tomonini differensiallash bilan ko‘rsatish mumkin.(buni bajarishni o‘quvchiga havola etamiz).

2-misol. $\int\left(\frac{1}{2\sqrt{x}}-\frac{1}{3\sqrt[3]{x^2}}\right)dx$ integralni hisoblang.

Yechish. Manfiy daraja xossasidan, hamda 4) xossadan foydalanib, jadvaldagi 1) formulaga asosan,

$$\int\left(\frac{1}{2\sqrt{x}}-\frac{1}{3\sqrt[3]{x^2}}\right)dx=\int\left(\frac{x^{-\frac{1}{2}}}{2}-\frac{x^{-\frac{2}{3}}}{3}\right)dx=\frac{1}{2}\int x^{-\frac{1}{2}}dx-\frac{1}{3}\int x^{\frac{2}{3}}dx=$$

$$=\frac{1}{2}\cdot\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}-\frac{1}{3}\cdot\frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1}+C=\frac{1}{2}\cdot\frac{\sqrt{x}}{\frac{1}{2}}-\frac{1}{3}\frac{\sqrt[3]{x}}{\frac{1}{3}}+C=\sqrt{x}-\sqrt[3]{x}+C$$

bo'ladi.

3-misol. $\int \frac{3\,dx}{\sin^2 x \cos^2 x}$ integralni hisoblang.

Yechish. $\sin^2 x + \cos^2 x = 1$ ayniyatdan hamda integralning 3) va 4) hossalaridan foydalanib hisoblaymiz:

$$\int \frac{3dx}{\sin^2 x \cos^2 x} = 3\int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx = 3\int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + 3$$

$$\int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx = 3\int \frac{1}{\cos^2 x} dx + 3\int \frac{1}{\sin^2 x} dx = 3(tgx - ctgx) + C.$$

4-misol. $\int \frac{dx}{\sqrt{5 - x^2}}$ integralni hisoblang.

Yechish. Jadvaldagi 9) formulaga asosan,

$$\int \frac{dx}{\sqrt{5 - x^2}} = \int \frac{dx}{\sqrt{(\sqrt{5})^2 - x^2}} = \arcsin \frac{x}{\sqrt{5}} + C.$$

Mustaqil bajarish uchun topshiriqlar

1. $\int \frac{5x^8 + 6}{x^4} dx$; 2. $\int \left(\frac{4}{x^2} - \frac{5}{x^3} \right) dx$; 3. $\int \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt[4]{x}} \right) dx$ 4. $\int \frac{x^2 + \sqrt{x} + 3dx}{\sqrt[3]{x^2}} dx$;

5. $\int e^x \left(1 - \frac{e^{-x}}{x^3} \right) dx$; 6. $\int \left(5^x + \frac{5^{-x}}{\sqrt{x}} \right) dx$; 7. $\int \frac{5 + 3tg^2 x}{\sin^2 x} dx$; 8. $\int \left(\frac{4}{9 + x^2} - \frac{5}{\sqrt{4 - x^2}} \right) dx$

9. $\int \frac{dx}{x^2 - 49}$; 10. $\int \frac{dx}{x^2 + 16}$; 11. $\int \left(\frac{5}{\sqrt{9 - x^2}} - \frac{3}{\sqrt{x^2 + 3}} \right) dx$; 12. $\int \left(\frac{7}{x^2 + 7} - \frac{6}{x^2 - 3} \right) dx.$

2-**mavzu. Aniqmas integralda integrallash usullari**

Reja

1. **<u>O‘zgaruvchini almashtirish</u>.**
2. **Bo‘laklab integrallash.**

1. <u>O‘zgaruvchini almashtirish</u>. Ko‘p hollarda yangi o‘zgaruvchi kiritish bilan integralni hisoblash, jadval integraliga keltiriladi. Bunda $\varphi(x)=t$ almashtirish olinib, bunda t yangi o‘zgaruvchi bo‘lib, o‘zgaruvchini almashtirish formulasi

$$\int f[\varphi(x)]\varphi'(x)dx=\int f(t)dt$$

ko‘rinishda bo‘ladi.

O‘zgaruvchini almashtirish usuliga bir necha misollar qaraymiz

1-misol. $\int (3x+1)^7\,dx$ integralni hisoblang.

Yechish. $3x+1=t$ deb $3dx=dt$ yoki $dx=\dfrac{dt}{3}$ ekanligini hisoblasak,

$$\int (3x+1)^7\,dx=\int t^7\,\frac{dt}{3}=\frac{1}{3}\cdot\frac{t^8}{8}+C=\frac{t^8}{24}+C=\frac{(3x+1)^8}{24}+C$$

bo’ladi.

2-misol. $\int \sqrt[3]{1+x^2}\;x\,dx$ integralni hisoblang.

Yechish. $1+x^2=t$ o‘zgaruvchi bilan almashtiramiz. Bu holda $2xdx=dt$ yoki $xdx=\dfrac{dt}{2}$ bo‘lib,

$$\int \sqrt[3]{1+x^2}\cdot xdx = \int \sqrt[3]{t}\cdot\frac{dt}{2} = \frac{1}{2}\int t^{\frac{1}{3}}dt = \frac{1}{2}\cdot\frac{t^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{8}t\sqrt[3]{t} + C = \frac{3}{8}(1+x^2)\sqrt[3]{1+x^2} + C$$

bo'ladi.

3-misol. $\int \cos mxdx$ integralni hisoblang.

Yechish. Bunda $dx = \frac{1}{m}d(mx)$ o'zgartirish olib,

$$\int \cos mxdx = \frac{1}{m}\int \cos mxd(mx) = \frac{1}{m}\sin mx + C$$

natijaga ega bo'lamiz. Bunday integrallashga ***<u>bevosita integrallash</u>*** deb ataladi. CHunki $mx = t$ bilan o'zgaruvchini almashtirib ham shu natijaga kelish mumkin edi. Yuqoridagi integralda o'zgaruvchini almashtirib o'tirmasdan uni fikrda bajardik.

4-misol. $\int (\ln x)^3 \frac{dx}{x}$ integralni hisoblang.

Yechish. $\ln x = t$ bilan yangi o'zgaruvchini almashtirib, $\frac{dx}{x} = dt$ ekanligini hisobga olsak,

$$\int (\ln x)^3 \frac{dx}{x} = \int t^3\cdot dt = \frac{t^4}{4} + C = \frac{(\ln x)^4}{4} + C$$

bo'ladi.

5-misol. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ integralni hisoblang.

Yechish. $x = t^2$ bilan yangi o'zgaruvchi kiritamiz oxirgi tenglikdan differensial topib, $dx = 2tdt$ bo'lganligi uchun,

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = \int \frac{\sin t}{t}2tdt = 2\int \sin tdt = -2\cos t + C = -2\cos\sqrt{x} + C$$

bo'ladi. —

6-misol. $\int e^{\sin x} \cdot \cos xdx$ integralni hisoblang.

Yechish. $\cos xdx = d(\sin x)$ ni hisobga olib,

$$\int e^{\sin x} \cdot \cos xdx = \int e^{\sin x} d(\sin x) = e^{\sin x} + C$$

natijaga kelamiz.

Shunday qilib, oddiy hollarda

$$xdx = \frac{1}{2}d(x^2),\quad \cos xdx = d(\sin x),\quad \frac{dx}{x} = d(\ln x),\quad dx = \frac{1}{a}(ax+b), \ldots.$$

tengliklardan foydalanib, o'zgaruvchini almashtirishni fikrda bajarib, bevosita integrallash ham mumkin.

2. Bo'laklab integrallash. Bo'laklab integrallash usuli differensial hisobning ikkita funksiya ko'paytmasi differensiali formulasiga asoslangan.

Ma'lumki, $d(uv) = udv + vdu,$ bundan $udv = d(uv) - vdu.$ Oxirgi tenglikni integrallab,

$$\int udv = \int d(uv) - \int vdu = uv - \int vdu$$

natijaga ega bo'lamiz. Shunday qilib,

$$\int udv = uv - \int vdu \qquad (1)$$

formulani hosil qildik. (1) formulaga ***bo'laklab integrallash*** formulasi deyiladi.

Bu formula yordamida berilgan $\int udv$ integraldan ikkinchi $\int vdu$ integralga o'tiladi. Demak, bo'laklab integrallashni qo'llash natijasida hosil bo‘lgan ikkinchi integral, berilgan integralga nisbatan soddaroq yoki jadval integrali bo‘lgandagina bu usulni qo‘llash maqsadga muvofiqdir. Bu maqsadga integral ostidagi ifodani u va dv ko‘paytuvchilarga qulay bo‘laklab olish natijasida erishish mukmin. Berilgan integral ostidagi ifodaning bir qismini u va qolgan qismini dv deb olgandan keyin (1) formuladan foydalanish uchun v va du larni aniqlash kerak bo‘ladi. du ni topish uchun u ning differensiali topilib, v ni topish uchun esa dv ifodani integralaymiz, bunda integral ixtiyoriy o‘zgarmas C ga bog‘liq bo‘lib, uning istalgan bir qiymatini xususiy holda $C=0$ ni olish mumkin.

Shunday qilib, integral ostidagi ifodaning bir qismini u deb olishda u differensiallash bilan soddalashadigan, qolgan qismi dv bo‘lib, qiyinchiliksiz integrallanadigan bo‘lishi kerak.

Bo‘laklab integrallash formulasi ko‘proq:

1) $\int p(x)e^{ax}dx,\ \int p(x)\sin mxdx,\ \int p(x)\cos axdx\quad va$
2) $\int p(x)\ln xdx, \int p(x)\arcsin xdx, \int p(x)\arccos xdx, \int p(x)arctgxdx,$
$\int p(x)arcctgxdx$

(bularda $p(x)$ biror darajali ko‘phad) ko‘rinishdagi integrallarni hisoblashda ishlatiladi. Bu integrallarni hisoblashda 1) guruh integrallarda u uchun $p(x)$ ko‘phad, qolgan qismi dv uchun olinib, 2) guruh integrallarda u uchun mos ravishda

$$\ln x,\ \arcsin x,\ \arccos x,\quad arctgx,\quad arcctgx \text{ lar},$$

qolgan qismi dv uchun olinadi.

Bo‘laklab integrallashga bir necha misollar qaraymiz.

1-misol. $\int x\cos xdx$ integralni hisoblang.

Yechish. Integral ostidagi ifodani $u=x,\ dv=\cos xdx$ deb

Bo‘laklasak, $du = dx, \quad v = \int \cos x dx = \sin x$ bo‘lib, (1) formulaga asosan,

$$\int \underset{u}{x} \underset{dv}{\cos x dx} = \underset{u}{x} \underset{v}{\sin x} - \int \underset{v}{\sin x} \underset{du}{dx} = x \sin x + \cos x + C$$

natijaga ega bo‘lamiz.

Bu integralda (1) formuladan foydalanish natijasida ikkinchi integral $\int \sin x \, dx$ hosil bo‘ldi, bu jadval integrali bo‘lganligi uchun osongina topildi.

2-misol. $\int x^2 e^{3x} dx$ integralni hisoblang.

Yechish. YUqorida eslatilganidek $u = x^2, \; dv = e^{3x} dx$ ko‘rinishda bo‘laklab olsak,

$$du = 2x dx, \quad v = \int e^{3x} dx = \frac{1}{3} \int e^{3x} d(3x) = \frac{1}{3} e^{3x}$$

hosil bo‘ladi. (1) formulaga asosan

$$\int x^2 e^{3x} dx = x^2 \frac{1}{3} e^{3x} - \int \frac{1}{3} e^{3x} 2x dx = \frac{x^2}{3} e^{3x} - \frac{2}{3} \int x e^{3x} dx$$

bo‘ladi. Oxirgi hosil bo‘lgan integral berilgan integralga nisbatan soddalashdi (berilgan integralda x ning 2- darajasi, ikkinchisida buning darajasi bittaga kamaydi). Keyingi integralda yana (1) formulani qo‘laymiz.

$$\int x e^{3x} dx = \left| \begin{matrix} u = x, \quad du = dx \\ dv = e^{3x}, \; v = \frac{1}{3} e^{3x} \end{matrix} \right| =$$

$$= x \frac{1}{3} e^{3x} - \frac{1}{3} \int e^{3x} dx = \frac{x}{3} e^{3x} - \frac{1}{3} \cdot \frac{1}{3} e^{3x} + C_1 = \frac{1}{3} e^{3x} \left(x - \frac{1}{3} \right) + C_1.$$

Shunday qilib, natijada

$$\int x^2e^{3x}dx=\frac{x^2}{3}e^{3x}-\frac{2}{3}\int xe^{3x}dx=\frac{x^2}{3}e^{3x}-\frac{2}{3}\left[\frac{1}{3}e^{3x}\left(x-\frac{1}{3}\right)+C_1\right]=$$

$$=\frac{e^{3x}}{3}\left(x^2-\frac{2x}{3}+\frac{2}{9}\right)+C$$

hosil bo‘ladi.

3-misol. $\int x^3\cos 2xdx$ integralni hisoblang.

Yechish. Yuqorida eslatilganidek emas, teskarisini ya’ni $u=\cos 2x,\quad dv=x^3dx$ bo‘laklab olaylik; bu holda

$du=-2\sin 2xdx,\quad v=\dfrac{x^4}{4}$ bo‘lib (1) formuladan foydalangandan

keyin

$$\int x^3\cos 2xdx=\cos 2x\cdot\frac{x^4}{4}+\int\frac{x^4}{4}\cdot 2\sin 2xdx=-\frac{x^4}{4}\cos 2x+$$

$$+\frac{1}{2}\int x^4\sin 2x\,dx$$

ifoda hosil bo‘ladi. Keyingi $\int x^4\sin 2xdx$ integral berilgan $\int x^3\cos 2xdx$ integralga nisbatan murakkabroqdir(x ning darajasi bittaga ortdi). Demak, bunday bo‘laklab olish maqsadga muvofiq emas, ya’ni $u=x^3,\quad dv=\cos 2xdx$ deb olish kerak edi. (Bu integralni hisoblashni o‘quvchiga havola qilamiz).

4-misol. $\int\arccos xdx$ integralni hisoblang.

Yechish.

$$\int \arccos x dx = \left| \begin{matrix} u = \arccos x, & du = -\dfrac{1}{\sqrt{1-x^2}dx} \\ dv = dx, & v = x \end{matrix} \right| = x\arccos x + \int \frac{xdx}{\sqrt{1-x^2}} =$$

$$= \left| \begin{matrix} 1 - x^2 = t \\ -2xdx = dt \\ xdx = -\dfrac{dt}{2} \end{matrix} \right| = x\arccos x + \int \frac{-\frac{dt}{2}}{\sqrt{t}} = x\arccos x - \frac{1}{2}\int t^{-\frac{1}{2}}dt = x\arccos x - \frac{1}{2}\cdot\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} +$$

$$+ C = x\arccos x - \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = x\arccos x - \sqrt{1-x^2} + C.$$

Bu integralda bir marta bo‘laklab integrallagandan keyingi hosil bo‘lgan integralda o‘zgaruvchini almashtirish usulidan foydalanib integralladik. Integrallash usullarini qo‘llashda o‘zgaruvchini almashtirganda yoki bo‘laklab integrallaganda yozuvda tartib bo‘lishi uchun yuqoridagi integralni hisoblangandagidek yozishga odat qilishni tavsiya etiladi.

5-misol. $J = \int e^x \cos x dx$ integralni hisoblang.

Yechish. Bo‘laklab integrallasak

$$J = \int e^x \cos x dx = \left| \begin{matrix} u = \cos x, \ du = -\sin x dx \\ dv = e^x dx, \ v = e^x \end{matrix} \right| = e^x \cos x + \int e^x \sin x dx,$$

hosil bo‘ladi.

Keyingi integral, berilgan integral bilan o‘xshashdek tuyuladi, lekin oxirgi integralda bo‘laklab integrallash formulasini ikkinchi marta qo‘llash bilan quyidagiga ega bo‘lamiz:

$$\int e^x \sin x dx = \left| \begin{matrix} u = \sin x, & du = \cos x dx \\ dv = e^x dx, & v = e^x \end{matrix} \right| = e^x \sin x - \int e^x \cos x dx$$

Shunday qilib,

$$J = e^x \cos x + e^x \sin x - J$$

J hisoblanishi kerak bo'lgan integralga nisbatan oddiy chiziqli tenglamaga keldik. Oxirgi tenglamadan

$$2J = e^x \cos x + e^x \sin x \qquad \text{ёки} \qquad J = \frac{1}{2} e^x (\cos x + \sin x) + C$$

natijaga ega bo'lamiz.

Mustaqil bajarish uchun topshiriqlar

Ushbu integrallarni hisoblang.

1. $\int \sin 3x dx$. 2. $\int \frac{2x-7}{x^2-7x+5} dx$.

3. $\int e^{-x^3} x^2 dx$. 4. $\int \frac{x^2}{1+x^3} dx$. 5. $\int (e^x + e^{-\frac{x}{2}}) dx$. 6. $\int \frac{dx}{\sin^2 5x}$.

7. $\int (5-2x)^9 dx$. 8. $\int \sqrt[3]{7-3x} dx$. 9. $\int \frac{1}{\sqrt[3]{6-x}} dx$. 10. $\int \cos(1-3x) dx$.

11. $\int \sin(5x+7) dx$. 12. $\int \frac{2x}{x^2+1} dx$. 13. $\int 3^{\sqrt{x}} \frac{dx}{\sqrt{x}}$. 14. $\int \frac{\sqrt{1+\ln x}}{x} dx$.

15. $\int \frac{1-3\sin x}{\cos^2 x} dx$ 16. $\int \frac{dx}{\sqrt{1-5x}}$ 17. $\int x^2 \sin x dx$ 18. $\int x \ln x dx$ 19. $\int arctgx dx.$

20. $\int \arcsin x dx$. 21. $\int x^2 e^{2x} dx$. 22. $\int e^x \sin x dx$. 23. $\int (x+3) \sin x dx$.

24. $\int x \cos\left(\frac{x}{2}\right) dx$. 25. $\int x arctgx dx.$ 26. $\int \frac{dx}{(3x+1)^3}$. 27. $\int \frac{2x+5}{x^2+5x+6} dx$.

3-mavzu. Ratsional va irratsional funksiyalarni integrallash

Reja

1. Ratsional kasr funksiyalarni integrallash:

1). To‘g‘ri va noto‘g‘ri kasr ratsional funksiyalar haqida;

2). To‘g‘ri kasr ratsional funsiyalarni sodda kasrlar ko‘rinishida ifodalash va ularni integrallash;

3). To‘g‘ri kasr ratsional funsiyalarni sodda kasrlar ko‘rinishida ifodalash.

2. Ayrim irratsional funksiyalarni integrallash.

1. Ratsional kasr funksiyalarni integrallash.

1). To‘g‘ri va noto‘g‘ri kasr ratsional funksiyalar haqida. YUqorida ko‘rsatilgan integrallash usullari yordamida hamma integrallarni hisoblash mumkin deb bo‘lmaydi.

Shunday funksiyalar sinflari borki, ular uchun muayyan usullardan foydalanib ularni jadval integrallariga yoki integrallash usullaridan foydalanish uchun qulay holga keltirish mumkin, shunday funksiya sinflaridan ayrimlarini qaraymiz.

Ma’lumki, har qanday ratsional funksiyani ushbu ko‘rinishida ifodalash mumkin, ya’ni

$$\frac{Q(x)}{P(x)}=\frac{b_0x^m+b_1x^{m-1}+\ldots+b_m}{a_0x^n+a_1x^{n-1}+\ldots a_n}.$$

Curatdagi ko‘phadning darajasi maxrajdagi ko‘phad darajasidan kichik, ya’ni $m<n$ bo‘lsa, berilgan kasrga ***to‘g‘ri kasr ratsional*** funksiya deyiladi. Suratdagi ko‘phadning darajasi $m\geq n$ bo‘lsa, ***noto‘g‘ri kasr ratsional funksiya*** deyiladi. Kasr noto‘g‘ri kasr ratsional funksiya bo‘lsa, suratni maxrajga, ko‘phadni ko‘phadga bo‘lish qoidasiga asosan bo‘lib, uning butun qismini ajratib, uni butun va to‘g‘ri kasr ratsional funksiyaga keltirish mumkin.

Masalan, $\frac{x^3+3x^2+3x+1}{x^2-x}$ noto‘g‘ri kasr ratsional funksiyani, x^3+3x^2+3x+1 ko’phadni x^2-x ko’phadga bo‘lib,

$$\frac{x^3+3x^2+3x+1}{x^2-x}=x+4+\frac{7x+1}{x^2-x}$$

ko‘rinishda yozish mumkin.

Umumiy holda, $\frac{Q(x)}{P(x)}$ noto‘g‘ri kasr ratsional funksiya bo‘lsa, uni

$$\frac{Q(x)}{P(x)}=T(x)+\frac{R(x)}{P(x)}$$

shakldа ifodalash mumkin, bu erda $T(x)$ butun ratsional funksiya, $\frac{R(x)}{P(x)}$ to‘g‘ri ratsional kasr funksiyadan iborat. $T(x)$ funksiyani osongina integrallash mumkin.

Shunday qilib, noto‘g‘ri kasr ratsional funksiyani integrallashni, $\frac{R(x)}{P(x)}$ to‘g‘ri kasr ratsional funksiyani integrallashga keltiriladi.

2). To‘g‘ri kasr ratsional funsiyalarni sodda kasrlar ko‘rinishida ifodalash va ularni integrallash

1) $\frac{A}{x-a}$; 2) $\frac{A}{(x-a)^k}$ ($k>1$ *бутун сон*); 3) $\frac{Ax+B}{x^2+px+q}$; ($\frac{p^2}{4}-q<0$ ya’ni, kvadrat uch had haqiqiy ildizga ega emas);

4) $\frac{Ax+B}{(x^2+px+q)^n}$ ($n>1$ butun son, $\frac{p^2}{4}-q<0$) ratsional to‘g‘ri kasrlarga ***sodda kasr ratsional funksiyalar*** deyiladi. (A, B, p, q, a - haqiqiy sonlar).

Birinchi ikki xildagi funksiyalarni osongina integrallash mumkin, ya’ni,

1) $\int \frac{A}{x-a}dx = A\ln|x-a| + C,$

2) $\int \frac{A}{(x-a)^k} = A\int (x-a)^{-k}d(x-a) = A\frac{(x-a)^{-k+1}}{-k+1} + C = \frac{A}{1-k}\cdot\frac{1}{(x-a)^{k-1}} + C$

bo'ladi. Endi ushbu

3) $\int \frac{Ax+B}{x^2+px+q}dx$

integralni hisoblaymiz.

Oldin xususiy hol $\int \frac{1}{x^2+px+q}dx$ integralni qaraylik. x^2+px+q dan to'la kvadrat ajratib, $x+\frac{p}{2}=t$ almashtirishdan keyin quyidagini hosil qilamiz:

$$\int \frac{1}{x^2+px+q}dx = \int \frac{1}{(x+\frac{p}{2})^2+q-\frac{p^2}{4}}dx = \left| \begin{matrix} x+\frac{p}{2}=t \\ dx=dt \end{matrix} \right| = \int \frac{dt}{(t^2+a^2)},$$

bu erda $a=\sqrt{q-\frac{p^2}{4}}$. Oxirgi integralda 8) jadval integralidan foydalanib,

$$\int \frac{1}{x^2+px+q}dx = \frac{1}{a}arctg\frac{t}{a} + C = \frac{2}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}} + C \qquad (2)$$

natijani hosil qilamiz.

Endi $\int \frac{Ax+B}{x^2+px+q}dx$ integralni hisoblaymiz.

$$Ax+B=(2x+p)\frac{A}{2}-\frac{Ap}{2}+B$$

shakl o'zgartirishdan foydalanib, integralni quyidagicha yozamiz.

$$\int \frac{Ax+B}{x^2+px+q}dx = \int \frac{(2x+p)\frac{A}{2} - \frac{Ap}{2} + B}{x^2+px+q}dx =$$

$$= \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx + \left(B - \frac{Ap}{2}\right)\int \frac{1}{x^2+px+q}dx.$$

Oxirgi tenglikning o‘ng tomonidagi birinchi integral

$$\int \frac{2x+p}{x^2+px+q}dx = \int \frac{d(x^2+px+q)}{x^2+px+q} = \ln\left|x^2+px+q\right| + C_1$$

bo‘lib, ikkinchi integral (2) formulaga asosan,

$$\int \frac{dx}{x^2+px+q} = \frac{2}{\sqrt{4q-p^2}} arctg \frac{2x+p}{\sqrt{4q-p^2}} + C_2 .$$

SHunday qilib,

$$\int \frac{Ax+B}{x^2+px+q}dx = \frac{A}{2}\ln\left|x^2+px+q\right| + \frac{2B-Ap}{\sqrt{4q-p^2}} arctg \frac{2x+p}{\sqrt{4q-p^2}} + C$$

natijaga ega bo‘lamiz.

Bir necha misollar qaraymiz.

1-misol. $\int \frac{x^4}{x^2+9}dx$ integralni hisoblang.

Yechish. Integral ostidagi funksiya noto‘g‘ri kasr ratsional funksiyadan iborat. Uning butun qismini ajratamiz:

$$
\begin{array}{r|l}
-\; x^4 & x^2+9 \\
\cline{2-2}
x^4+9x^2 & x^2-9 \\
\cline{1-1}
-\; -9x^2 & \\
-9x^2-81 & \\
\cline{1-1}
81 &
\end{array}
$$

Demak, $\dfrac{x^4}{x^2+9}=x^2-9+\dfrac{81}{x^2+9}$

bo’ladi.

Shunday qilib,

$$\int \frac{x^4}{x^2+9}dx=$$

$$=\int\left(x^2-9+\frac{81}{x^2+9}\right)dx=\frac{x^3}{3}-9x+81\frac{1}{3}arctg\frac{x}{3}+C=\frac{x^3}{3}-9x+27arctg\frac{x}{3}+C.$$

2-misol. $\int\dfrac{x+3}{x^2-8x+25}dx$ integralni hisoblang.

Yechish. Maxrajdagi kvadrat uch haddan to‘la kvadrat ajratamiz: $x^2-8x+25=x^2-8x+16-16+25=(x-4)^2+9,$ hamda $x-4=t,\ dx=dt$ almashtirish kiritib, quyidagini hosil qilamiz:

$$\int\frac{x+3}{x^2-8x+25}dx=\int\frac{t+4+3}{t^2+9}dt=\int\frac{tdt}{t^2+9}+7\int\frac{dt}{t^2+9}=\frac{1}{2}\int\frac{2tdt}{t^2+9}+7\int\frac{dt}{t^2+3^2}=$$

$$=\frac{1}{2}\ln\left|t^2+9\right|+\frac{7}{3}arctg\frac{t}{3}+C=\frac{1}{2}\ln(x^2-8x+25)+\frac{7}{3}arctg\frac{x-4}{3}+C.$$

3. To‘g‘ri kasr ratsional funsiyalarni sodda kasrlar ko‘rinishida ifodalash.

$\dfrac{R(x)}{P(x)}$ to‘g‘ri kasr ratsional funksiyaning maxrajini

$$P(x)=(x-a)^r\cdot(x-b)^s.....(x^2+2px+q)^t\cdot(x^2+2kx+\ell)^m.....,$$

ko‘rinishda ifodalash mumkin bo‘lsa, bu funksiyani yagona

$$\frac{R(x)}{P(x)}=\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+...+\frac{A_r}{(x-a)^r}+\frac{B_1}{(x-b)}+...+\frac{B_s}{(x-b)^s}+...+\frac{M_1x+N_1}{x^2+2px+q}+...+$$
$$+\frac{M_tx+N_t}{(x^2+2px+q)^t}+\frac{F_1x+E_1}{\left(x^2+2kx+\ell\right)}+...+\frac{F_mx+E_m}{\left(x^2+2kx+\ell\right)^m}+... \quad (1)$$

ko‘rinishda yozish mumkin. Bunda $r,s,\ .t,m,$ musbat butun sonlar, $a,\ b,\ p,\ q,\ k,\ \ell$, haqiqiy sonlar.

$A_1,\ A_2,....\ A_r,\ B_1,...,B_s,\ M_1,\ N_1,.....,\ M_t,\ N_t,....$ lar ayrim haqiqiy sonlar. (1) tenglikka to‘g‘ri ratsional funksiyaning ***<u>sodda kasrlar orqali yoyilmasi</u>*** deyiladi.

(1) yoyilmadagi $A_1,\ A_2,....\ A_r,\ M_1,\ N_1,.....,\ M_t,\ N_t,....$ koeffitsientlarni topish uchun uni $P(x)$ ga ko‘paytiramiz. $R(x)$ ko‘phad bilan (1) yoyilmaning o‘ng tomonida hosil bo‘lgan ko‘phad o‘zaro teng bo‘lishi uchun bir xil darajali x lar koeffitsientlari o‘zaro teng bo‘lishi kerak. Bir xil darajali x lar koeffitsientlarini tenglashtirib $A_1,\ A_2,....\ A_r,\ .....,M_1,\ N_1,....,$ noma’lum koeffitsentlarga nisbatan chiziqli tenglamalar sistemasini hosil qilamiz. Bu tenglamalar sistemasini echib aniqmas koeffitsientlarni topamiz.

Ratsional funkiya yoyilmasidagi noma’lum koeffitsientlarni bunday usul bilan topishga ***<u>noma’lum koeffitsientlar usuli</u>*** deyiladi.

Bu usulni bir necha misollarda qaraymiz.

1-misol. $\dfrac{2x-1}{x^2-5x+6}$ ratsional funksiyani sodda kasrlar yoyilmasi ko‘rinishida yozing.

Yechish. Maxrajni $x^2-5x+6=(x-3)(x-2)$ chiziqli ko‘paytuvchilarga ajratib, (1) formulaga asosan, qo‘yidagicha yozamiz:

$$\frac{2x-1}{x^2-5x+6}=\frac{A_1}{x-3}+\frac{A_2}{x-2}.$$

Oxirgi tenglikni x^2-5x+6 ga ko‘paytirib $2x-1=A_1(x-2)+A_2(x-3)$ *ёки* $2x-1=(A_1+A_2)x-2A_1-3A_2$ tenglikni hosil qilamiz. Bir xil darajali x lar koeffitsientlarini tenglashtirib

$$\begin{cases} A_1+A_2=2, \\ -2A_1-3A_2=-1 \end{cases}$$

A_1 va A_2 noma’lumlarga nisbatan, chiziqli tenglamalar sistemasini hosil qildik. Bundan $A_1=5,\ A_2=-3$ bo‘ladi. Shunday qilib,

$$\frac{2x-1}{x^2-5x+6}=\frac{5}{x-3}-\frac{3}{x-2}$$

hosil bo‘ldi.

2-misol. $\frac{x^2-1}{x(x^2+1)^2}$ ratsional kasrni, sodda kasrlar yig‘indisi ko‘rinishida yozing.

Yechish. (1) formulaga asosan, quyidagicha ifodalash mumkin:

$$\frac{x^2-1}{x(x^2+1)^2}=\frac{A_1}{x}+\frac{M_1x+N_1}{x^2+1}+\frac{M_2x+N_2}{(x^2+1)^2}.$$

Oxirgi tenglikni $x(x^2+1)^2$ ga ko‘paytirib quyidagini hosil qilamiz:

$$x^2-1=A_1(x^2+)^2+(M_1x+N_1)x(x^2+1)+(M_2x+N_2)x \quad \textit{yoki}$$

$$x^2-1=(A_1+M_1)x^4+N_1x^3+(2A_1+M_1+M_2)x^2+(N_1+N_2)x+A_1$$

$x^0, x^1,\ x^2,\ x^3,\ x^4$ larning koeffitsientlarini tenglashtirib

$$
\begin{aligned}
&x^4 \div 2A_1 + M_1 = 0,\\
&x^3 \div N_1 = 0,\\
&x^2 \div 2A_1 + M_1 + M_2 = 1,\\
&x^1 \div N_1 + N_2 = 0,\\
&x^0 \div A_1 = -1
\end{aligned}
$$

chiziqli tenglamalar sistemasini hosil qilamiz. Bu sistemadan $N_1 = 0,\ N_2 = 0,\ A_1 = -1,\ M_1 = 1,\ M_2 = 2$ bo‘ladi. Shuning uchun yoyilma quyidagicha

$$\frac{x^2-1}{x(x^2+1)^2} = -\frac{1}{x} + \frac{x}{x^2+1} + \frac{2x}{(x^2+1)^2}$$

bo‘ladi.

Endi bir necha integrallarni hisoblaymiz.

3-misol. $\int \frac{dx}{(x-2)(x-3)}$ integralni hisoblang.

Yechish. $\frac{1}{(x-2)(x-3)}$ funksiyani $\left(\frac{-1}{x-2} + \frac{1}{x-3}\right)$ ikkita sodda kasrning ayirmasi ko‘rinishda yozish mumkin. Shuning uchun,

$$\int \frac{1}{(x-2)(x-3)}dx = -\int\left(\frac{1}{x-2} - \frac{1}{x-3}\right)dx = -\ln|x-2| + \ln|x-3| + C = \ln\left(\frac{x-3}{x-2}\right) + C$$

bo‘ladi.

4-misol. $\int \frac{x-2}{x^3+2x^2}dx$ integralni hisoblang.

Yechish. $\frac{x-2}{x^2(x+2)}$ integral ostidagi to‘g‘ri ratsional funksiyani sodda kasrlar yig‘indisi shaklida yozamiz:

$$\frac{x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

oxirgi tenglikni $x^2(x+2)$ ga ko‘paytirib,

$$x-2 = Ax(x+2) + B(x+2) + Cx^2 \quad yoki \quad x-2 = (A+C)x^2 + (2A+B)x + 2B$$

tenglikka ega bo‘lamiz.

Endi x ning bir xil darajalilari koeffitsientlarini tenglashtirsak, quyidagi tenglamalar sistemasi hosil bo‘ladi.

$$\begin{array}{c|l} x^2 & A+C=0 \\ x^1 & 2A+B=1 \\ x^0 & 2B=-2 \end{array} \quad \text{bundan} \quad \begin{array}{c} A+C=0 \\ 2A+B=1 \\ B=-1 \end{array} \quad \begin{array}{c} A=1 \\ C=-1 \end{array}$$

bo‘ladi. Shunday qilib,

$$\frac{x-2}{x^2(x+2)} = \frac{1}{x} + \frac{-1}{x^2} + \frac{-1}{x+2}$$

Natijada

$$\int \frac{x-2}{x^3+2x^2}dx = \int\left(\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x+2}\right)dx = \ln|x| + \frac{1}{x} - \ln|x+2| + C = \frac{1}{x} + \ln\left|\frac{x}{x+2}\right| + C$$

bo‘ladi.

2. Ayrim irratsional funksiyalarni integrallash. ***<u>Irratsional funksiyalarni integrallash</u>*** ko‘p hollarda o‘zgaruvchini almashtirish bilan ratsional funksiyalarni integrallashga keltiriladi. Bunday irratsional funksiyalarning ayrimlarini qaraymiz.

1. $\int x^m(a+bx^n)^p$ **ko’rinishdagi integral**ni ќisoblash talab etilsin,

bunda m, n, p ratsional sonlar, a va b lar no‘ldan farqli o‘zgarmaslar.

1) p butun son bo‘lsa, Nyuton binomi bo‘yicha yoyish bilan integrallanadi;

2) $\frac{m+1}{n}$ butun bo‘lsa, $a+bx^n = t^s$ almashtirish orqali

ratsionallashtiriladi, bunda s p kasrning maxraji;

3) $\frac{m+1}{n} + p$ butun bo‘lsa, $ax^{-n} + b = t^s$ almashtirish olinib,

ratsional funksiyaga keltiriladi.

1-misol. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}} dx$ integralni hisoblang.

Yechish. Integralni $\int x^{-2/3}(1+x^{1/3})dx$ ko‘rinishida yozib,

$$m = -\frac{2}{3}, \; n = \frac{1}{3}, \; p = \frac{1}{2}, \; \frac{m+1}{n} = \frac{-2/3+1}{1/3} = \frac{1/3}{1/3} = 1$$

bo‘lganligi uchun $(1+x^{1/3}) = t^2$ almashtirish olsak,

$$x^{1/3} = t^2 - 1, \; \frac{1}{3}x^{-2/3}\,dx = 2tdt, \qquad x^{-2/3}dx = 6tdt$$

bo‘ladi. Bularni berilgan integralga qo‘ysak,

$$\int x^{-2/3}(1+x^{1/3})^{1/2}dx = \int t \cdot 6tdt = 6\int t^2 dt = 6 \cdot \frac{t^3}{3} + C = 2(1+\sqrt[3]{x})^{3/2} + C$$

bo‘ladi.

2-misol. $\int \frac{dx}{\sqrt[4]{1+x^4}}$ integralni hisoblang.

Yechish. $\int \frac{dx}{\sqrt[4]{1+x^4}} = \int x^0 (1+x^4)^{-1/4} dx$, $m = 0$, $n = 4$, $p = -\frac{1}{4}$

$\frac{m+1}{n} + p = \frac{1}{4} - \frac{1}{4} = 0$(butun) bo‘lganligi uchun $x^{-4} + 1 = t^4$ almashtirish olsak,

$x = (t^4 - 1)^{-1/4}$, $dx = -t^3 (t^4 - 1)^{-5/4} dt$, $\sqrt[4]{1+x^4} = t(t^4 - 1)^{-1/4}$

bo‘ladi.Demak,

$$\int \frac{dx}{\sqrt[4]{1+x^4}} = -\int \frac{t^2 dt}{t^4 - 1} = \frac{1}{4}\int \left(\frac{1}{1+t} - \frac{1}{t-1} \right) dt - \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{4}\ln\frac{|t+1|}{t-1} - \frac{1}{2} arctgt + C,$$

$t = \sqrt[4]{1+x^4}$ bo‘lganligi uchun,

$$\int \frac{dx}{\sqrt[4]{1+x^4}} = \frac{1}{4}\ln\frac{\left|\sqrt[4]{1+x^4}\right| + 1}{\sqrt[4]{1+x^4} - 1} - \frac{1}{2} arctg\sqrt[2]{1+x^4} + 1 + C$$

bo‘ladi.

2. $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ **ko‘rinishdagi integralni qaraymiz.**

Bunday ko‘rinishdagi ifodalarni integrallash kvadrat uch haddan to‘la kvadrat ajratish bilan $\int \frac{du}{\sqrt{a^2 - u^2}}$ yoki $\int \frac{du}{\sqrt{a^2 + u^2}}$ jadval integrallaridan biriga keltiriladi.

3-misol. $\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$ integralni hisoblang.

Yechish. $x^2 + 2x + 5 = x^2 + 2x + 1 + 4 = (x+1)^2 + 4$ to‘la kvadrat ajratib,

$x + 1 = u$ desak,

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{du}{\sqrt{u^2+4}} = \ln\left|u+\sqrt{u^2+4}\right| + C = \ln\left|(x+1)+\sqrt{x^2+2x+5}\right| + C$$

bo‘ladi.

3) $\int \frac{dx}{(x+\alpha)\sqrt{ax^2+bx+c}}$ ko‘rinishdagi integral, $\frac{1}{x+\alpha} = t$ almashtirish orqali **2.** ko‘rinishdagi integralga keltiriladi.

4-misol. $\int \frac{dx}{(x+1)\sqrt{x^2+2x+2}}$ integralni hisoblang.

Yechish. $\frac{1}{x+1} = t$ bilan almashtirsak,

$$t(x+1)=1,\ tx+t=1,\ tx=1,\ x=\frac{1-t}{t},\ dx=-\frac{1}{t^2}dt$$

bo‘lib,

$$\int \frac{dx}{(x+1)\sqrt{x^2+2x+2}} = \int \frac{t\left(\frac{-1}{t^2}\right)dt}{\sqrt{\left(\frac{1-t}{t}\right)^2 + 2\frac{1-t}{t} + 2}} = -\int \frac{dt}{t\sqrt{\frac{1-2t+t^2}{t^2} + \frac{2-2t}{t} + 2}} =$$

$$= -\int \frac{dt}{\sqrt{1-2t+t^2+2t}} = -\int \frac{dt}{\sqrt{t^2+1}}$$

bu jadval integraldir. (Oxirgi integralni mustaqil bajarishni o‘quvchiga havola qilamiz).

Mustaqil bajarish uchun topshiriqlar

1. Ushbu ratsional funksiyalarni integrallang.

1) $\int \frac{x^3}{x-3}dx$; 2) $\int \frac{x^4}{x^2+25}dx$; 3) $\int \frac{x-5}{(x-2)(x+4)}dx$; 4) $\int \frac{2x+9}{x^2+x+2}dx$;

5) $\int \frac{x+2}{x^3-2x^2}dx$; 6) $\int \frac{5x-1}{2x^2+x-3}dx$; 7) $\int \frac{2x+5}{(x-4)(x+5)}dx$; 8) $\int \frac{6x-7}{x^3-4x^2+4x}dx$.

2. Ushbu irratsional funksiyalarni integrallang.

1) $\int \frac{2x+1}{\sqrt{x^2-4x+1}}dx$; 2) $\int \frac{2x-3}{\sqrt{8-2x-x^2}}dx$; 3) $\int \frac{7x-5}{\sqrt{\sqrt{5+2x-x^2}}}dx$;

4) $\int \frac{dx}{\sqrt[3]{(2x+1)^2}-\sqrt{2x+1}}$; 5) $\int \frac{\sqrt{x}+3}{x\left(\sqrt{x}+\sqrt[3]{x}\right)}dx$.

4-mavzu. Trigonometrik funksiyalarni integrallash

Reja

1. Harxil argumentli sinus va kosinuslar ko‘paytmalari shaklidagi funksiyalarni integrallash.

2. $\int \sin^m x \cos^n x dx$ **ko‘rinishdagi integrallarni hisoblash.**

3. Aniqmas integral haqida yakuniy mulohazalar.

1. Har xil argumentli sinus va kosinuslar ko‘paytmalari shaklidagi funksiyalarni integrallash.

$$\int \sin mx \cos nx dx, \quad \int \sin mx \sin nx dx, \quad \int \cos mx \cos nx dx \qquad (1)$$

ko‘rinishdagi integrallarni ҡisoblaymiz. Maktab kursidan ma’lum bo‘lgan trigonometrik funksiyalar ko‘paytmasini, yig‘indiga keltirish

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha+\beta)], \quad \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)],$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

formulalardan foydalanib, (1) ko‘rinishdagi integrallarni

$$\int \sin ax dx, \quad \int \cos bx dx$$

integrallardan biriga keltirib itegrallanadi.

1-misol. $\int \sin 2x \cos 7x dx$ integralni hisoblang.

Yechish. Yuqoridagi formulalarning birinchisidan

$$\sin 2x\cos 7x=\frac{1}{2}\left[\sin(2x+7x)+\sin(2x-7x)\right]=\frac{1}{2}(\sin 9x\sin 5x),$$

$$\int \sin 2x\cos 7xdx=\frac{1}{2}\int(\sin 9x-\sin 5x)dx=\frac{1}{2}\int \sin 9xdx-\frac{1}{2}\int \sin 5xdx=$$

$$=\frac{1}{2}\cdot\frac{1}{9}(-\cos 9x)-\frac{1}{2}\cdot\frac{1}{5}(-\cos 5x)+C=\frac{1}{10}\cos 5x-\frac{1}{18}\cos 9x+C.$$

natijaga ega bo‘lamiz.

2-misol. $\int \cos 7x\cos 3xdx,\quad \int \sin 4x\sin 2xdx,\quad \int \sin 5x\cos 3x$ integrallarni mustaqil hisoblang.

2. $\int \sin^m x\cos^n xdx$ **ko‘rinishdagi integrallarni hisoblash.** Bunda m,n lar butun sonlar. Xususiy hollarda m yoki n sonlardan birontasi 0 ga teng bo‘lishi ham mumkin.

1) m yoki n sonlardan bittasi toq bo‘lsin. Bu holda integral ratsional funksiyalarni integrallashga keltiriladi. Bunda integrallash mohiyati quyidagi misollardan tushunarli bo‘ladi.

3-misol. $\int \sin^3 x\cos^4 xdx$ integralni hisoblang.

Yechish. $\sin xdx=-d(\cos x)$ va $\sin^2 x=1-\cos^2 x$ ekanligini hamda $\cos x=z$ almashtirish kiritib, quyidagini hosil qilamiz:

$$\int \sin^3 x\cos^4 xdx=\int \sin^2 x\cos^4 x\sin xdx=\int(1-\cos^2 x)\cos^4 x(-d\cos x)=$$

$$=-\int(1-z^2)z^4dz=-\int(z^4-z^6)dz=-\frac{z^5}{5}+\frac{z^7}{7}+C=-\frac{\cos^5 x}{5}+\frac{\cos^7 x}{7}+C.$$

4-misol. $\int \frac{\sin^3 x}{\cos^2 x}dx$ integralni hisoblang.

Yechish. $\sin x dx = -d\cos x$ bo‘lgani uchun, $t = \cos x$ almashtirish olsak,

$$\int \frac{\sin^3 x}{\cos^2 x} dx = \int \frac{\sin^2 x \sin x dx}{\cos^2 x} = \int \frac{1-\cos^2 x}{\cos^2 x} \sin x dx = \int \frac{1-t^2}{t^2}(-dt) =$$

$$= -\int \frac{1}{t^2} dt + \int dt = \frac{1}{t} + t + C = \frac{1}{\cos x} + \cos x + C$$

bo’ladi.

Bu usuldan m va n sonlardan bittasi toq va musbat boshqasi ixtiyoriy haqiqiy son bo‘lganda ham foydalanish mumkin.

2). Endi m va n sonlar ikkalasi ham toq yoki juft va musbat bo‘lsin. Bunday hollarda

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}, \ \sin x \cos x = \frac{1}{2}\sin 2x$$

formulalardan foydalanib, darajalarni pasaytirib, integrallanadi.

6-misol. $\int \sin^2 x\, dx$ integralni hisoblang.

Yechish. Bu integralni izoḱlarsiz ḱisoblaymiz:

$$\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx = \int \frac{1}{2} dx - \int \frac{\cos 2x}{2} dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

7-misol. $\int \sin^2 x \cos^4 x dx$ integralni hisoblang.

Yechish. Trigonometrik funksiyalarning darajalarini pasaytirish formulalaridan foydalanib, quyidagi natijaga kelamiz:

$$\int \sin^2 x \cos^4 x dx = \int \frac{1-\cos 2x}{2} \cdot \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{8}\int (1-\cos 2x)(1+\cos 2x)^2 dx =$$

$$= \frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x)dx = \frac{1}{8}\int \sin^2 2x dx + \frac{1}{8}\int (\sin^2 2x \cos 2x dx) =$$

$$= \frac{1}{16}\int (1-\cos 4x)dx + \frac{1}{16}\int \sin^2 2x d \sin 2x = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{16}\frac{\sin^3 2x}{3} + C =$$

$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.$$

3. *<u>Aniqmas integral haqida yakuniy mulohazalar</u>*. Biz yuqorida elementar funksiyalarni o‘z ichiga olgan muhim integrallash usullarini ko‘rdik. Lekin praktikada faqat shu usullardan aynan foydalanamiz degan fikr bo‘lmasligi kerak. Boshqacha qilib aytganda, integral ostidagi funksiyaning berilishiga qarab unga mos mulohazalardan foydalanish kerak. Masalan,

$$\int \frac{3x^2+6x-4}{x^3+3x^2-4x+5}dx = \int \frac{d(x^3+3x^2-4x+5)}{x^3+3x^2-4x+5} = \int \frac{dt}{t} = \ln|t| + C =$$
$$= \ln\left|x^3+3x^2-4x+5\right| + C$$

yoki $\int \frac{xdx}{x^2+3} = \frac{1}{2}\int \frac{2xdx}{x^2+3} = \frac{1}{2}\int \frac{d(x^2+3)}{x^3+3} = \frac{1}{2}\ln\left|x^2+3\right| + C$

integrallashni bajarish mumkin.

Juda ko‘p integrallarni hisoblashda ayrim xususiy usullardan foydalanib oldingi hisoblangan integrallarga keltiriladi. Shuning uchun praktikada integrallashda tayyor qo‘llanmalardan foydalanish ham mumkin. Integrallashning bayon etilishidan ma’lumki integrallash texnikasi differensiallashdan murakkabroqdir. Shuning uchun ham integrallashda shunday ko‘nikmalar kerakki, bunga ko‘p sondagi misollarni Yechish natijasida erishish mumkin.

Ma’lumki differensial hisobda istalgan elementar funksiyaning hosilasini topish mumkin edi va u yana elementar funksiyalar bilan ifodalanar edi. Integral hisobda esa masala boshqacharoq bo‘lib, ko‘plab

misollar keltirish mumkinki, integral $\left(e^{-x^2}, \frac{1}{\ln x}, \frac{\sin x}{x}, \frac{\cos x}{x}\right)$ ostidagi funksiyaning boshlang‘ich funksiyalari mavjud bo‘lishiga qaramasdan, ular elementar funksiyalar orqali ifodalanmaydi. Bunday integrallar yaxshi o‘rganilgan va ulardan amaliyotda foydalanish uchun tayyor jadvallar, grafiklar tuzilgan.

Mustaqil bajarish uchun topshiriqlar

Ushbu integrallarni hisoblang.

1. $\int \sin 3x \sin 7x dx$. 2. $\int \sin 5x \cos 3x dx$. 3. $\int \sin x \sin 3x dx$.
4. $\int \sin 3x \cos 2x dx$. 5. $\int \cos 4x \cos 2x dx$. 6. $\int \sin 3x \cos x dx$. .
7. $\int \sin x \cos^4 x dx$. 8. $\int \sin^3 x \cos^3 x dx$. 9. $\int \sin^2 5x dx$.

10. $\int \cos 7x \cos 3x dx$. 11. $\int \sin 4x \sin 2x dx$.

12. $\int \sin^2 x \cos^2 x dx$. 13. $\int \frac{\cos^5 x}{\sin^2 x} dx$. 14. $\int \frac{\cos^3 x dx}{\sin^2 x}$.

15. $\int tg^3 x dx$. 16. $\int ctg^3 x dx$. 17. $\int \frac{\sin^3 x + 1}{\cos^2 x} dx$.

18. $\int \sin^4 x dx$. 19. $\int \cos^4 x dx$. 20. $\int \sin^5 x dx$.

21. $\int \cos^5 x dx$. 22. $\int \sin^2 x \cos^3 x dx$. 23. $\int \sqrt[3]{cos^2 x} \sin^3 x \, dx$.

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